JOURNAL OF ALGEBRA 2, 15-37 (1965)

Tensor Products in Categories*

G. M. KELLY

Tulane University, Louisiana

Communicated by Saunders MacLane

Received May 4, 1964

1. INTRODUCTION

MacLane has pointed out (see [*3*], §24) that the various types of "category with extra structure" that have appeared in the literature, from pointed or preadditive categories to the graded differential categories used in the unpublished work of Eilenberg and Moore or in [*1*], may all be described by axioms like those for a category, except that $\mathrm{Hom}(A, B)$ is not a set but an object of a category $\mathscr{P}$ provided with a covariant bifunctor $\otimes : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$, and that composition is replaced by a morphism

$$\mathrm{Hom}(B, C) \otimes \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

in $\mathscr{P}$. Moreover, he has observed that, in all cases, $\otimes$ is possessed of at least the following properties: there are natural isomorphisms

$$\rho : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

$\theta : I \otimes A \to A$, $\tau : A \otimes I \to A$, $\mu : A \otimes B \to B \otimes A$ (where I is a fixed "ground object" of $\mathscr{P}$) which are *coherent* in the sense that any natural automorphism defined by their repeated use, such as

$$A \otimes B \xrightarrow[\tau^{-1}\otimes 1]{} (A \otimes I) \otimes B \xrightarrow[\rho]{} A \otimes (I \otimes B) \xrightarrow[1\otimes\theta]{} A \otimes B,$$

is to be the identity. He has further shown in [*4*] that the apparently infinite set of conditions imposed by the requirement of coherence in fact follows from a finite number among them, which are happily precisely those one would pick as most basic. Finally, he has also shown [*3*, *4*] that, in an *abelian* category $\mathscr{P}$, a functor which, in addition to having the above properties, is additive and right exact, is an acceptable generalization of the tensor product in the category of modules over a commutative ring.

* The work on this paper was partially supported by NSF Grant Number GP 41.

For a general category $\mathscr{P}$, what further properties should $\otimes$ have, in order that we may reasonably call it a "tensor product"? Being associative, commutative, and having a two-sided identity, to within coherent natural isomorphisms, is not really enough: the direct sum $\oplus$ in the category of abelian groups has these properties, but we should not want to call it a tensor product. If we insist that $\otimes$ have a *coadjoint*, we at one stroke rule out such examples as this, and automatically obtain the condition of right-exactness in the abelian case; moreover, in the examples of categories with extra structure, the functors $\otimes$ that arise do in fact have coadjoints. Further, there is a widespread folk-feeling that, whatever else a tensor product is, it ought to be the adjoint of something one might call a Hom-functor; for an "internal" tensor product such as we are considering, this cannot be the usual Hom-functor to sets, but must be defined on $\mathscr{P}^* \times \mathscr{P}$ and take values in $\mathscr{P}$ itself; and it seems reasonable to call such a thing an "internal Hom-functor" if, when composed with some functor $F : \mathscr{P} \to \mathscr{S}$, where $\mathscr{S}$ is the category of sets, it yields the usual Hom-functor to sets. We might think of requiring F to be *faithful*, (meaning that it maps the set of morphisms from A to B injectively into the set of morphisms from FA to FB), but to do so would in fact exclude important examples.

Notationally it is convenient to denote the set of morphisms from A to B in any category $\mathscr{P}$ by $\mathscr{P}(A, B)$; and then it is natural to use the symbol $\mathscr{P}$ itself to denote the usual Hom-functor to sets, writing $\mathscr{P}(f, g)$ for the map $h \to ghf$ of $\mathscr{P}(A, B)$ into $\mathscr{P}(\bar{A}, \bar{B})$ induced by $f : \bar{A} \to A$ and $g : B \to \bar{B}$. In this notation an internal Hom-functor is a functor $\mathbf{P} : \mathscr{P}^* \times \mathscr{P} \to \mathscr{P}$ (where $\mathscr{P}^*$ denotes the category dual to $\mathscr{P}$) such that $F\mathbf{P} = \mathscr{P} : \mathscr{P}^* \times \mathscr{P} \to \mathscr{S}$.

Now, so long as $\otimes$ has a left identity I, its coadjoint $\mathbf{P}$, if any, must be an internal Hom-functor; for we have natural isomorphisms

$$\mathscr{P}(A, B) \cong \mathscr{P}(I \otimes A, B) \cong \mathscr{P}(I, \mathbf{P}(A, B)),$$

and if we define $F : \mathscr{P} \to \mathscr{S}$ by $F = \mathscr{P}(I, -)$, we have $F\mathbf{P} \cong \mathscr{P}$, which is not (as we shall see) essentially different from $F\mathbf{P} = \mathscr{P}$.

Once a functor $\otimes : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$ is the adjoint of an internal Hom-functor $\mathbf{P}$, it has some claim to be called a tensor product, whether or not it is associative, commutative, or has left or right identities; and we shall so call it. (It would be going too far to let $\otimes$ merely have a coadjoint, without insisting that this be an internal Hom-functor: to do so would allow such obviously pathological examples as the functor $\otimes$ defined by $A \otimes B = A$ for all A, B.) It then becomes a problem to examine what hypotheses on $\mathscr{P}$, F, $\mathbf{P}$, and the adjunction, would be equivalent to the associativity and so on of $\otimes$.

This approach has various advantages. For one thing the necessary hypotheses on $\mathscr{P}$, $\mathbf{P}$ etc. turn out to be very simple and natural, and their individual effects may be seen in isolation. For another, we can investigate the extent to which $\otimes$, ρ, θ, τ, μ are determined by $\mathscr{P}$, F, and $\mathbf{P}$ alone, and prove various uniqueness results. Finally the matter of categories with extra structure is not quite as simple as it appears above: for instance, how does one define the identity morphisms? In each individual case that has arisen, it has been clear enough what to do; but it is not obvious what the right formulation is for a general $\mathscr{P}$. This gets sorted out in the language we develop to discuss our tensor products. We first introduce $\mathscr{P}$-categories, and the corresponding notions of $\mathscr{P}$-functor and $\mathscr{P}$-adjoint, which do not need a tensor product for their definition, and which include pointed categories and preadditive categories, but not graded differential categories. If $\mathscr{P}$ itself is a $\mathscr{P}$-category it has an internal Hom-functor $\mathbf{P}$, and we can discuss the adjoint $\otimes$, if any, of $\mathbf{P}$. We can then define the notion of $\mathbf{P}$-category, which uses $\otimes$ in its definition, which includes graded differential categories, and which in certain cases coincides with $\mathscr{P}$-category. (The notion of $\mathbf{P}$-category, together with a study of the relation of the tensor product to the direct product, as well as the question of commutativity, we defer to the second part of this paper, to be published separately.)[1]

We end this introduction with some remarks on our notation. We use brackets no more than is necessary for clarity or emphasis, and in particular often write fx for the value of a function f for the argument x. We also use juxtaposition to denote composition of functions, of morphisms, of functors, and of natural transformations, so long as these are represented by single symbols; otherwise we use a dot, and write e.g. $\alpha \cdot S\lambda \cdot \beta$ for the composition of β, $S\lambda$, and α. If $T : \mathscr{A} \to \mathscr{B}$ is a functor we denote by T_{AB} the map $f \to Tf$ of $\mathscr{A}(A, B)$ into $\mathscr{B}(TA, TB)$, but usually abbreviate this to

$$T : \mathscr{A}(A, B) \to \mathscr{B}(TA, TB).$$

If we think of A and B as variable, this last T is a natural transformation, and we often so regard it. If S, $T : \mathscr{A} \to \mathscr{B}$ are functors and $\lambda : T \to S$ a natural transformation, we write λ_A for the morphism $TA \to SA$ given by λ, but usually drop the subscript A. If T and S are functors of many variables we may speak of $\lambda : T(A, B, C) \to S(A, B, C)$ as being natural in any or all of A, B, C; if we just say "λ is natural", we mean natural in all three. We note that, if T, $S : \mathscr{B} \to \mathscr{A}$ are functors and $\lambda : T \to S$ a natural transformation, $\mathscr{A}(1, \lambda) : \mathscr{A}(A, TB) \to \mathscr{A}(A, SB)$ is natural in A and B; while if T and S are fixed objects of $\mathscr{A}$ and $\lambda : T \to S$ a morphism, $\mathscr{A}(1, \lambda) : \mathscr{A}(A, T) \to \mathscr{A}(A, S)$ is still natural in A. We also note that, if T, $S : \mathscr{A} \to \mathscr{B}$ and if $\lambda : T \to S$ is

[1] *Note added in proof:* The projected second part of this paper will not now appear, being superseded by a forthcoming joint paper of Eilenberg and the author. The material in this first part remains entirely relevant.

a natural transformation, its naturality can be expressed by the commutativity of

$$\begin{tikzcd} \mathscr{A}(A, A') \arrow[r, "T"] \arrow[d, "S"'] & \mathscr{B}(TA, TA') \arrow[d, "\mathscr{B}(1,\lambda)"] \\ \mathscr{B}(SA, SA') \arrow[r, "\mathscr{B}(\lambda,1)"'] & \mathscr{B}(TA, SA'). \end{tikzcd}$$

Finally, if $T, S : \mathscr{A} \to \mathscr{B}$ and $\lambda : T \to S$, and if also $P : \mathscr{C} \to \mathscr{A}$ and $Q : \mathscr{B} \to \mathscr{D}$, we denote the natural transformations resulting from substitution by $Q\lambda : QT \to QS$ and $\lambda_P : TP \to SP$, which is consistent with $(Q\lambda)_A = Q\lambda_A$, $(\lambda_P)_C = \lambda_{PC}$; also we often write λ for λ_P (but never λ for $Q\lambda$) in informal contexts.

2. Adjoints

We recall in this section the essentials of the theory of adjoints, due to Kan [*2*], since we shall have constantly to refer to it in detail.

The basic proposition is:

Theorem 1. *If $S \in \mathscr{A}$ and $B \in \mathscr{B}$ are fixed objects and $T : \mathscr{A} \to \mathscr{B}$ a functor, then any transformation a: $\mathscr{A}(S, A) \to \mathscr{B}(B, TA)$, natural in A, factorizes as*

$$\mathscr{A}(S, A) \xrightarrow[T]{} \mathscr{B}(TS, TA) \xrightarrow[\mathscr{B}(\beta,1)]{} \mathscr{B}(B, TA)$$

for a unique morphism $\beta : B \to TS$.

β is unique because, putting $A = S$ and applying a and $\mathscr{B}(\beta, 1)T$ to $1 = 1_S$, we get $a1 = \mathscr{B}(\beta, 1)T1 = \beta$. Then for any $f \in \mathscr{A}(S, A)$ we have $f = \mathscr{A}(1, f)1_S$ and so, by the naturality of a, $af = a\mathscr{A}(1, f)1 = \mathscr{B}(1, Tf)a1 = \mathscr{B}(1, Tf)\beta = Tf \cdot \beta = \mathscr{B}(\beta, 1)Tf$.

Remark. There is an obvious dual proposition referring to a natural transformation $a : \mathscr{A}(A, S) \to \mathscr{B}(TA, B)$.

Corollary. *Any natural transformation $a : \mathscr{A}(S, A) \to \mathscr{A}(B, A)$ is $\mathscr{A}(\beta, 1)$ for some $\beta : B \to S$, and a is an isomorphism if and only if β is.*

Now suppose that $T : \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \to \mathscr{A}$ are functors, and that $a : \mathscr{A}(SB, A) \to \mathscr{B}(B, TA)$ is natural in A. Then for each fixed B we get a morphism $\beta = \beta_B : B \to TSB$ with $a = \mathscr{B}(\beta, 1)T$.

Theorem 2. *a is natural in B too if and only if $\beta : 1 \to TS$ is a natural transformation.*

The "if" part is obvious. Suppose a is natural in B; then so is

$$\mathscr{B}(B, B') \xrightarrow[S]{} \mathscr{A}(SB, SB') \xrightarrow[a]{} \mathscr{B}(B, TSB'),$$

so that, by the corollary to Theorem 1, $aS = \mathscr{B}(1, \bar{\beta})$ where $\bar{\beta} = aS1$. But $aS1 = a1 = \beta$, so that $aS = \mathscr{B}(1, \beta)$. Writing a as $\mathscr{B}(\beta, 1)T$, we now have $\mathscr{B}(\beta, 1)TS = \mathscr{B}(1, \beta)$, which says precisely (see the last paragraph of Section 1) that $\beta : 1 \to TS$ is natural.

If now $T : \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \to A$ are functors we say S is an *adjoint* of T and T is a *coadjoint* of S, written $S \dashv T$, if there is a natural *isomorphism* (called the adjunction)

$$a : \mathscr{A}(SB, A) \to \mathscr{B}(B, TA).$$

Let b be the inverse of a. By Theorem 2 there are natural transformations

$$\alpha : ST \to 1, \beta : 1 \to TS$$

such that a and b factorize as

$$a = \mathscr{B}(\beta, 1)T : \mathscr{A}(SB, A) \to \mathscr{B}(TSB, TA) \to \mathscr{B}(B, TA), \tag{1}$$

$$b = \mathscr{A}(1, \alpha)S : \mathscr{B}(B, TA) \to \mathscr{A}(SB, STA) \to \mathscr{A}(SB, A). \tag{2}$$

Alternatively, we might have started with T, S, α, and β, and defined a and b by (1) and (2); and we ask when they would be mutually inverse. The map $ba : \mathscr{A}(SB, A) \to \mathscr{A}(SB, A)$ is, by Theorem 1 cor., $\mathscr{A}(\lambda, 1)$ where $\lambda = ba1 = b\beta = \alpha \cdot S\beta$: or more properly $\alpha_S \cdot S\beta$. Doing the same for ab, we find that ba and ab are both 1 if and only if

$$\alpha_S \cdot S\beta = 1 \qquad \text{and} \qquad T\alpha \cdot \beta_T = 1. \tag{3}$$

In the proof of Theorem 2 we found $aS = \mathscr{B}(1, \beta)$; now that a is an isomorphism this gives

$$S = b\mathscr{B}(1, \beta) : \mathscr{B}(B, B') \to \mathscr{B}(B, TSB') \to \mathscr{A}(SB, SB'), \tag{4}$$

and similarly

$$T = a\mathscr{A}(\alpha, 1) : \mathscr{A}(A, A') \to \mathscr{A}(STA, A') \to \mathscr{B}(TA, TA'). \tag{5}$$

Thus S and T are determined *on morphisms* by their values on objects and by a, b, α, β.

THEOREM 3. *Given a functor $T : \mathscr{A} \to \mathscr{B}$, a function S from the objects of $\mathscr{B}$ to those of $\mathscr{A}$, and an isomorphism $a : \mathscr{A}(SB, A) \to \mathscr{B}(B, TA)$ natural in A for each B; then there is exactly one way of defining S on morphisms that makes it a functor and renders a natural in B.*

Here a determines β as before, and we are forced to define S on morphisms by $S = b\mathscr{B}(1, \beta)$ as in (4) (where $b = a^{-1}$), showing the uniqueness. The defining equation for S is equivalent to $aS = \mathscr{B}(1, \beta)$ or, by (1), $\mathscr{B}(\beta, 1)TS = \mathscr{B}(1, \beta)$: once we have shown S to be a functor, this shows $\beta : 1 \to TS$ to be natural, and hence a to be natural in B. Applying $\mathscr{B}(\beta, 1)TS = \mathscr{B}(1, \beta)$ to $f \in \mathscr{B}(B, B')$ gives $TSf \cdot \beta = \beta f$ as a defining equation for Sf. Since $T1 \cdot \beta = \beta 1$ and $T(Sf \cdot Sg)\beta = TSf \cdot TSg \cdot \beta = TSf \cdot \beta \cdot g = \beta fg$, we have $S1 = 1$ and $S(fg) = Sf \cdot Sg$, as required.

Now let $T, T' : \mathscr{A} \to \mathscr{B}$ and $S, S' : \mathscr{B} \to \mathscr{A}$, with $S \dashv T$ and a, b, α, β as before, and also $S' \dashv T'$ with a', b', α', β'.

THEOREM 4. *To each natural transformation $\lambda : T \to T'$ corresponds exactly one natural transformation $\lambda^* : S' \to S$, called the adjoint of λ, making the diagram*

$$\begin{tikzcd}
\mathscr{A}(SB, A) \arrow[r, "a"] \arrow[d, "\mathscr{A}(\lambda^*, 1)"'] & \mathscr{B}(B, TA) \arrow[d, "\mathscr{B}(1, \lambda)"] \\
\mathscr{A}(S'B, A) \arrow[r, "a'"'] & \mathscr{B}(B, T'A)
\end{tikzcd}$$

commutative; if $\mu : T' \to T''$ is a further natural transformation with $S'' \dashv T''$, then $(\mu\lambda)^* = \lambda^*\mu^*$, and $1^* = 1$.

COROLLARY. *Any two adjoints of T are naturally isomorphic.*

Since $b'\mathscr{B}(1, \lambda)a : \mathscr{A}(SB, A) \to \mathscr{A}(S'B, A)$ is natural, the existence and uniqueness of λ^* follow from Theorem 2. The rest of the theorem follows from the uniqueness, and the corollary is then obvious.

Note that by Theorem 1 the commutativity of the above diagram is fully equivalent to the equality of $a'\mathscr{A}(\lambda^*, 1)1_{SB}$ and $\mathscr{B}(1, \lambda)a1_{SB}$, that is, to

$$T'\lambda^* \cdot \beta' = \lambda_S \cdot \beta. \tag{6}$$

Dually we find the equivalent condition

$$\alpha \cdot \lambda^*_T = \alpha' \cdot S'\lambda. \tag{7}$$

Also by applying $\mathscr{A}(\lambda^*, 1)$ and $b'\mathscr{B}(1, \lambda)a$ to 1_{SB}, we get an explicit formula for λ^*:

$$\lambda^* = \alpha'_S \cdot S'\lambda_S \cdot S'\beta. \tag{8}$$

Now let $T : \mathscr{C}^* \times \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \times \mathscr{C} \to \mathscr{A}$ be bifunctors. We say S is an adjoint of T and T a coadjoint of S, $S \dashv T$, if there is a natural isomorphism

$$a : \mathscr{A}(S(B, C), A) \to \mathscr{B}(B, T(C, A)).$$

In this case $S(-, C)$ is an adjoint of $T(C, -)$ for each fixed C. Moreover, since any $h : C' \to C$ induces a natural transformation

$$T(h, 1) : T(C, -) \to T(C', -),$$

the naturality of a in C shows that $S(-, h)$ is $T(h, -)^*$. It is clear from this that if T were given as a bifunctor and $S(B, C)$ given merely as a functor of B for each fixed C, with the isomorphism a natural in A and B, then S could be extended to a bifunctor in just one way so as to make a natural in C also. Further it is clear—since λ^* and μ^* commute when λ and μ do so—that Theorem 4 and its corollary have an immediate extension to the present case.

We define $b = a^{-1}$, α, β as before; but α and β now depend upon C:

$$\alpha : S(T(C, A), C) \to A, \ \beta : B \to T(C, S(B, C)).$$

Equations (6) and (7) show the naturality of a in C to be equivalent to the commutativity of either of the following diagrams:

$$\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[d, "\beta"'] & T(C, S(B, C)) \arrow[d, "T(h,1)"] \\
T(C', S(B, C')) \arrow[r, "T(1,S(1,h))"'] & T(C', S(B, C)),
\end{tikzcd} \tag{9}$$

$$\begin{tikzcd}
A & S(T(C, A), C) \arrow[l, "\alpha"'] \\
S(T(C', A), C') \arrow[u, "\alpha"] & S(T(C, A), C'). \arrow[l, "S(T(h,1),1)"] \arrow[u, "S(1,h)"']
\end{tikzcd} \tag{10}$$

Situations such as these, where C appears, say, twice in the range (with opposite variances), and not at all in the domain, of a transformation, are quite common. It will be convenient in such cases to express the commutativity of diagrams such as (9) and (10) by saying: α and β are *natural* in C.

3. $\mathscr{P}$-Categories and $\mathscr{P}$-Functors

By a *semiconcrete* category $(\mathscr{P}, F)$ (or just $\mathscr{P}$ by abuse of language) we mean the pair consisting of a category $\mathscr{P}$ and a functor $F : \mathscr{P} \to \mathscr{S}$ where $\mathscr{S}$ is the category of sets. We call $(\mathscr{P}, F)$ *concrete* if F is faithful.

Given a semiconcrete category $\mathscr{P}$ we define a *$\mathscr{P}$-category* $(\mathscr{A}, \mathbf{A})$ to be a category $\mathscr{A}$ together with a functor $\mathbf{A} : \mathscr{A}^* \times \mathscr{A} \to \mathscr{P}$ with

$$F\mathbf{A} = \mathscr{A} : \mathscr{A}^* \times \mathscr{A} \to \mathscr{S}.$$

(The reader will easily verify that, if we had only a natural isomorphism $F\mathbf{A} \cong \mathscr{A}$, we could replace $\mathscr{A}$ by an isomorphic category and $\mathbf{A}$ by its transform under this isomorphism, and recover actual equality $F\mathbf{A} = \mathscr{A}$. The same is true even when $\mathscr{A}$ coincides with $\mathscr{P}$, provided that we then replace F too by its transform.)

By a $\mathscr{P}$-functor $(T, \mathbf{T}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$, where $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$ are $\mathscr{P}$-categories, is meant a functor $T : \mathscr{A} \to \mathscr{B}$ together with a natural transformation $\mathbf{T} : \mathbf{A}(A, A') \to \mathbf{B}(TA, TA')$ such that

$$F\mathbf{T} = T : \mathscr{A}(A, A') \to \mathscr{B}(TA, TA').$$

The composition of two $\mathscr{P}$-functors $(S, \mathbf{S})$ and $(T, \mathbf{T})$ is the $\mathscr{P}$-functor $(ST, \mathbf{ST})$.

If $(T, \mathbf{T})$ and $(S, \mathbf{S})$ are $\mathscr{P}$-functors from $(\mathscr{A}, \mathbf{A})$ to $(\mathscr{B}, \mathbf{B})$, a $\mathscr{P}$-natural transformation $\theta : (T, \mathbf{T}) \to (S, \mathbf{S})$ is a natural transformation $\theta : T \to S$ for which the diagram

$$\begin{tikzcd} \mathbf{A}(A, A') \arrow[r, "\mathbf{T}"] \arrow[d, "\mathbf{S}"'] & \mathbf{B}(TA, TA') \arrow[d, "\mathbf{B}(1,\theta)"] \\ \mathbf{B}(SA, SA') \arrow[r, "\mathbf{B}(\theta,1)"'] & \mathbf{B}(TA, SA') \end{tikzcd}$$

commutes. If $(L, \mathbf{L}) : (\mathscr{C}, \mathbf{C}) \to (\mathscr{A}, \mathbf{A})$ and $(M, \mathbf{M}) : (\mathscr{B}, \mathbf{B}) \to (\mathscr{D}, \mathbf{D})$ are also $\mathscr{P}$-functors, the reader will easily verify that $\theta_L : (TL, \mathbf{TL}) \to (SL, \mathbf{SL})$ and $M\theta : (MT, \mathbf{MT}) \to (MS, \mathbf{MS})$ are also $\mathscr{P}$-natural. Finally, if

$$\theta : (T, \mathbf{T}) \to (S, \mathbf{S})$$

is $\mathscr{P}$-natural and $\theta : T \to S$ is a natural *isomorphism*, then $\theta^{-1} : (S, \mathbf{S}) \to (T, \mathbf{T})$ is also $\mathscr{P}$-natural.

If $\mathscr{C}$ is a category in the ordinary sense, by a $\mathscr{P}$-$\mathscr{S}$-bifunctor (or more briefly a $\mathscr{P}$-bifunctor) $(\mathscr{A}, \mathbf{A}) \times \mathscr{C} \to (\mathscr{B}, \mathbf{B})$ we mean a bifunctor $T : \mathscr{A} \times \mathscr{C} \to \mathscr{B}$ together with a transformation

$$\mathbf{T} : \mathbf{A}(A, A') \to \mathbf{B}(T(A, C), T(A', C)),$$

natural in A, A', *and* C (see the last paragraph of §2), such that

$$F\mathbf{T} = T(-, C) : \mathscr{A}(A, A') \to \mathscr{B}(T(A, C), T(A', C)).$$

It comes to the same thing to say that we have a $\mathscr{P}$-functor in A for each fixed C, and that the morphisms of C induce $\mathscr{P}$-natural transformations.

Finally, by a $\mathscr{P}$-$\mathscr{P}$-bifunctor $(\mathscr{A}, \mathbf{A}) \times (\mathscr{C}, \mathbf{C}) \to (\mathscr{B}, \mathbf{B})$, where these are all $\mathscr{P}$-categories, we mean a functor $T : \mathscr{A} \times \mathscr{C} \to \mathscr{B}$ with two natural transformations

$$\mathbf{T} : \mathbf{A}(A, A') \to \mathbf{B}(T(A, C), T(A', C)),$$
$$\mathbf{t} : \mathbf{C}(C, C') \to \mathbf{B}(T(A, C), T(A, C')),$$

such that $F\mathbf{T} = T(-, C)$ and $F\mathbf{t} = T(A, -)$.

We generally allow such abuses of language as "$T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{P}$-functor", "$\theta : T \to S$ is $\mathscr{P}$-natural", when it is clear what $\mathbf{A}$, $\mathbf{B}$, $\mathbf{T}$, $\mathbf{S}$ are; but note that it really is possible for the one functor T to admit two distinct $\mathscr{P}$-functor structures $\mathbf{T}$ and $\mathbf{T}'$.

All these concepts may be more simply described when F is faithful. A $\mathscr{P}$-category is then just a category $\mathscr{A}$ with the assignment of an object $\mathbf{A}(A, B)$ of $\mathscr{P}$, with $F\mathbf{A}(A, B) = \mathscr{A}(A, B)$, to each pair A, B of objects of $\mathscr{A}$, subject to the requirement that the map $g \to hgf$ of $\mathscr{A}(A, B)$ into $\mathscr{A}(A', B')$, induced by morphisms $f \in \mathscr{A}(A', A)$ and $h \in \mathscr{A}(B, B')$, is the image under F of some morphism $\mathbf{A}(A, B) \to \mathbf{A}(A', B')$ in $\mathscr{P}$: the latter morphism is then automatically unique, and if we call it $\mathbf{A}(f, h)$, $\mathbf{A}$ is a functor. Taking $\mathscr{P}$ to be pointed sets or abelian groups, and F the forgetful functor, we regain the usual definitions of pointed and preadditive categories.

A $\mathscr{P}$-functor in the faithful case is just a functor $T : \mathscr{A} \to \mathscr{B}$ such that each $T : \mathscr{A}(A, A') \to \mathscr{B}(TA, TA')$ is the image under F of some morphism $\mathbf{T} : \mathbf{A}(A, A') \to \mathbf{B}(TA, TA')$; the latter is then automatically unique and natural. We recognize the definition of pointed and additive functors.

Lastly, in the faithful case the condition of $\mathscr{P}$-naturality is vacuous, coinciding with ordinary naturality: for the diagram expressing the $\mathscr{P}$-naturality of θ passes under the action of F to that expressing the mere naturality of θ.

4. $\mathscr{P}$-ADJOINTS

Let $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$ be $\mathscr{P}$-categories and $T : \mathscr{A} \to \mathscr{B}$, $S : \mathscr{B} \to \mathscr{A}$ ordinary functors. We say S and T are $\mathscr{P}$-adjoint, $S \dashv_{\mathscr{P}} T$, if there is a natural isomorphism (called a *$\mathscr{P}$-adjunction*)

$$\mathbf{a} : \mathbf{A}(SB, A) \to \mathbf{B}(B, TA).$$

It follows that $a = F\mathbf{a} : \mathscr{A}(SB, A) \to \mathscr{B}(B, TA)$ is an isomorphism, so that $S \dashv T$. Also a has inverse $b = F\mathbf{b}$, where $\mathbf{b} = \mathbf{a}^{-1}$; and a, b determine as usual $\alpha : ST \to 1$ and $\beta : 1 \to TS$.

THEOREM 5. *If $S \underset{\mathscr{P}}{\dashv} T$ the $\mathscr{P}$-adjunction* $\mathbf{a}$ *canonically endows S and T with $\mathscr{P}$-functor structures* $\mathbf{S}$ *and* $\mathbf{T}$, *with respect to which α and β are $\mathscr{P}$-natural*; *we then write* $(S, \mathbf{S}) \underset{\mathscr{P}}{\dashv} (T, \mathbf{T})$. *Moreover* $\mathbf{a}$ *and* $\mathbf{b}$ *factorize as*

$$\mathbf{a} = \mathbf{B}(\beta, 1)\mathbf{T} : \mathbf{A}(SB, A) \to \mathbf{B}(TSB, TA) \to \mathbf{B}(B, TA), \tag{11}$$

$$\mathbf{b} = \mathbf{A}(1, \alpha)\mathbf{S} : \mathbf{B}(B, TA) \to \mathbf{A}(SB, STA) \to \mathbf{A}(SB, A). \tag{12}$$

We define $\mathbf{T}$ and $\mathbf{S}$ by

$$\mathbf{T} = \mathbf{aA}(\alpha, 1) : \mathbf{A}(A, A') \to \mathbf{A}(STA, A') \to \mathbf{B}(TA, TA'), \tag{13}$$

$$\mathbf{S} = \mathbf{bB}(1, \beta) : \mathbf{B}(B, B') \to \mathbf{B}(B, TSB') \to \mathbf{A}(SB, SB'). \tag{14}$$

Then $F\mathbf{T} = a\mathscr{A}(\alpha, 1) = T, F\mathbf{S} = b\mathscr{B}(1, \beta) = S$, by (4) and (5); and $\mathbf{T}$ and $\mathbf{S}$ are clearly natural. Then (11) follows because

$$\begin{aligned} \mathbf{B}(\beta, 1)\mathbf{T} &= \mathbf{B}(\beta, 1)\mathbf{aA}(\alpha_S, 1) \\ &= \mathbf{aA}(S\beta, 1)\mathbf{A}(\alpha_S, 1) \qquad \text{by the naturality of } \mathbf{a} \\ &= \mathbf{a}, \qquad \text{since } \alpha_S \cdot S\beta = 1 \text{ by (3).} \end{aligned}$$

Equations (11) and (14) now give

$$\mathbf{B}(\beta, 1)\mathbf{TS} = \mathbf{aS} = \mathbf{abB}(1, \beta) = \mathbf{B}(1, \beta),$$

showing that β is $\mathscr{P}$-natural.

THEOREM 6. *Let* $(T, \mathbf{T}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ *and* $(S, \mathbf{S}) : (\mathscr{B}, \mathbf{B}) \to (\mathscr{A}, \mathbf{A})$ *be $\mathscr{P}$-functors, and let $a : \mathscr{A}(SB, A) \to \mathscr{B}(B, TA)$ be an ordinary adjunction, with $b = a^{-1}$, α, β as usual. Then if α and β are $\mathscr{P}$-natural with respect to* $\mathbf{T}$ *and* $\mathbf{S}$, *we have* $(S, \mathbf{S}) \underset{\mathscr{P}}{\dashv} (T, \mathbf{T})$ *by a $\mathscr{P}$-adjunction* $\mathbf{a}$ *with* $F\mathbf{a} = a$.

Define $\mathbf{a}$ and $\mathbf{b}$ by (11) and (12); then certainly $F\mathbf{a} = a$ and $F\mathbf{b} = b$. We have to show that $\mathbf{a}$ and $\mathbf{b}$ are inverse isomorphisms. We have

$$\begin{aligned} \mathbf{ba} &= \mathbf{A}(1, \alpha)\mathbf{SB}(\beta, 1)\mathbf{T} \\ &= \mathbf{A}(1, \alpha)\mathbf{A}(S\beta, 1)\mathbf{ST} \qquad \text{by the naturality of } \mathbf{S} \\ &= \mathbf{A}(S\beta, 1)\mathbf{A}(1, \alpha)\mathbf{ST} \\ &= \mathbf{A}(S\beta, 1)\mathbf{A}(\alpha_S, 1) \qquad \text{by the } \mathscr{P}\text{-naturality of } \alpha \\ &= 1, \qquad \text{because } \alpha_S \cdot S\beta = 1 \text{ by (3).} \end{aligned}$$

Finally we must check that the $\mathscr{P}$-functor structures that S and T receive from $\mathbf{a}$ coincide with those they already have; that is, verify (13) and (14). But since α is $\mathscr{P}$-natural,

$$\mathbf{A}(\alpha, 1) = \mathbf{A}(1, \alpha)\mathbf{ST}, = \mathbf{bT};$$

thus $\mathbf{T} = \mathbf{aA}(\alpha, 1)$, as required.

Now consider, with the same $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$, two $\mathscr{P}$-adjunctions $(S, \mathbf{S}) \underset{\mathscr{P}}{\dashv} (T, \mathbf{T})$ and $(S', \mathbf{S}') \underset{\mathscr{P}}{\dashv} (T', \mathbf{T}')$. Passing by F to the ordinary adjunctions $S \dashv T$, $S' \dashv T'$, consider a natural transformation $\lambda : T \to T'$ and its adjoint $\lambda^* : S' \to S$.

THEOREM 7. *The diagram*

$$\begin{tikzcd} \mathbf{A}(SB, A) \arrow[r, "\mathbf{a}"] \arrow[d, "\mathbf{A}(\lambda^*,1)"'] & \mathbf{B}(B, TA) \arrow[d, "\mathbf{B}(1,\lambda)"] \\ \mathbf{A}(S'B, A) \arrow[r, "\mathbf{a}'"'] & \mathbf{B}(B, T'A) \end{tikzcd}$$

commutes if and only if λ and λ^* are both $\mathscr{P}$-natural; and if either of them is $\mathscr{P}$-natural, so is the other.

Suppose λ is $\mathscr{P}$-natural. Then

$$\begin{aligned} \mathbf{B}(1, \lambda)\mathbf{a} &= \mathbf{B}(1, \lambda)\mathbf{B}(\beta, 1)\mathbf{T} \\ &= \mathbf{B}(\beta, 1)\mathbf{B}(1, \lambda)\mathbf{T} \\ &= \mathbf{B}(\beta, 1)\mathbf{B}(\lambda_S, 1)\mathbf{T}' \qquad \text{by the } \mathscr{P}\text{-naturality of } \lambda \\ &= \mathbf{B}(\lambda_S \cdot \beta, 1)\mathbf{T}' \\ &= \mathbf{B}(T'\lambda^* \cdot \beta', 1)\mathbf{T} \qquad \text{by (6)} \\ &= \mathbf{B}(\beta', 1)\mathbf{B}(T'\lambda^*, 1)\mathbf{T}' \\ &= \mathbf{B}(\beta', 1)\mathbf{T}'\mathbf{A}(\lambda^*, 1) \qquad \text{by the naturality of } \mathbf{T}' \\ &= \mathbf{a}'\mathbf{A}(\lambda^*, 1); \end{aligned}$$

thus the diagram commutes. Suppose the diagram commutes. Then

$$\begin{aligned} \mathbf{B}(1, \lambda)\mathbf{T} &= \mathbf{B}(1, \lambda)\mathbf{aA}(\alpha, 1) \\ &= \mathbf{a}'\mathbf{A}(\lambda^*_T, 1)\mathbf{A}(\alpha, 1) \qquad \text{by hypothesis} \\ &= \mathbf{a}'\mathbf{A}(\alpha \cdot \lambda^*_T, 1) \\ &= \mathbf{a}'\mathbf{A}(\alpha' \cdot S'\lambda, 1) \qquad \text{by (7)} \\ &= \mathbf{a}'\mathbf{A}(S'\lambda, 1)\mathbf{A}(\alpha', 1) \\ &= \mathbf{B}(\lambda, 1)\mathbf{a}'\mathbf{A}(\alpha', 1) \qquad \text{by the naturality of } \mathbf{a}' \\ &= \mathbf{B}(\lambda, 1)\mathbf{T}'; \end{aligned}$$

thus λ is $\mathscr{P}$-natural. That $\mathscr{P}$-naturality of λ implies that of λ^* now follows from the symmetry of the condition that the diagram should commute; or, alternatively, directly from (8).

The reader will easily verify that, if

$$(T, \mathbf{T}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B}), \quad (S, \mathbf{S}) : (\mathscr{B}, \mathbf{B}) \to (\mathscr{A}, \mathbf{A}),$$
$$(R, \mathbf{R}) : (\mathscr{B}, \mathbf{B}) \to (\mathscr{C}, \mathbf{C}), \quad (Q, \mathbf{Q}) : (\mathscr{C}, \mathbf{C}) \to (\mathscr{B}, \mathbf{B}),$$

are all $\mathscr{P}$-functors, and if $(S, \mathbf{S}) \dashv_{\mathscr{P}} (T, \mathbf{T})$ and $(Q, \mathbf{Q}) \dashv_{\mathscr{P}} (R, \mathbf{R})$, then $(SQ, \mathbf{SQ}) \dashv_{\mathscr{P}} (RT, \mathbf{RT})$.

Now let $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$ be $\mathscr{P}$-categories, $\mathscr{C}$ any category, and $T : \mathscr{C}^* \times \mathscr{A} \to \mathscr{B}$, $S : \mathscr{B} \times \mathscr{C} \to \mathscr{A}$ ordinary bifunctors. We say that S and T are $\mathscr{P}$-adjoint, $S \dashv_{\mathscr{P}} T$, if there is a natural isomorphism

$$\mathbf{a} : \mathbf{A}(S(B, C), A) \to \mathbf{B}(B, T(C, A)).$$

THEOREM 8. *If $S \dashv_{\mathscr{P}} T$, the $\mathscr{P}$-adjunction* $\mathbf{a}$ *canonically endows S and T with $\mathscr{P}$-bifunctor structures* $\mathbf{S} : \mathbf{B}(B, B') \to \mathbf{A}(S(B, C), S(B', C))$ *and* $\mathbf{T} : \mathbf{A}(A, A') \to \mathbf{B}(T(C, A), T(C, A'))$, *and we also write* $(S, \mathbf{S}) \dashv_{\mathscr{P}} (T, \mathbf{T})$. *Conversely, if T has a $\mathscr{P}$-bifunctor structure* $\mathbf{T}$ *and if* $(T(C, -), \mathbf{T})$ *has a $\mathscr{P}$-adjoint for each fixed C, then* $(T, \mathbf{T})$ *has a $\mathscr{P}$-adjoint* $(S, \mathbf{S})$.

This follows at once from Theorem 7.

Now suppose that $(S, \mathbf{S}) \dashv_{\mathscr{P}} (T, \mathbf{T})$ as in Theorem 8, and suppose that $\mathscr{C}$ too has a $\mathscr{P}$-category structure $\mathbf{C}$. Suppose further that T is a $\mathscr{P}$-$\mathscr{P}$-bifunctor, with $\mathbf{t} : \mathbf{C}(C, C') \to \mathbf{B}(T(C', A), T(C, A))$ giving the $\mathscr{P}$-functor structure for the argument C.

THEOREM 9. *There is then exactly one way of making S into a $\mathscr{P}$-$\mathscr{P}$-bifunctor by a $\mathscr{P}$-functor structure* $\mathbf{s}$: $\mathbf{C}(C, C') \to \mathbf{A}(S(B, C), S(B, C'))$ *which will make* $\beta : B \to T(C, S(B, C))$ *$\mathscr{P}$-natural in C as well as in B; and if S is given this structure,* $\alpha : S(T(C, A), C) \to A$ *is also $\mathscr{P}$-natural in C as well as in A.*

We must explain what we mean by "β is $\mathscr{P}$-natural in C"; this is a natural amalgamation of the concepts of $\mathscr{P}$-naturality and of naturality of β in C, and means the commutativity of

$$\begin{tikzcd}
\mathbf{C}(C, C') \arrow[r, "\mathbf{t}"] \arrow[d, "\mathbf{s}"'] & \mathbf{B}(T(C', S(B, C')), \quad T(C, S(B, C'))) \arrow[dd, "\mathbf{B}(\beta,1)"] \\
\mathbf{A}(S(B, C), S(B, C')) \arrow[d, "\mathbf{T}"'] & \\
\mathbf{B}(T(C, S(B, C)), T(C, S(B, C'))) \arrow[r, "\mathbf{B}(\beta,1)"'] & \mathbf{B}(B, T(C, S(B, C'))).
\end{tikzcd}$$

Since the composition of $\mathbf{B}(\beta, 1)$ and $\mathbf{T}$ in this diagram is $\mathbf{a}$, and since $\mathbf{b} = \mathbf{a}^{-1}$, the diagram says

$$\mathbf{s} = \mathbf{b}\mathbf{B}(\beta, 1)\mathbf{t}, \tag{15}$$

which shows that $\mathbf{s}$ is unique. Moreover $\mathbf{s}$ so defined is natural in C, C' and B; this is immediate from the naturality of $\mathbf{t}$, $\mathbf{B}$, β and $\mathbf{b}$, together with the following two propositions whose proofs the reader may provide:

1. If $\theta : A \to L(B, A, B)$ and $\phi : L(B, A, C) \to M(B, A, C)$ are natural (where L and M are contravariant in their first argument and covariant in the others), then $\phi\theta : A \to M(B, A, B)$ is natural.

2. If $\theta : A \to L(B, A, B)$ and $\psi : L(B, B, C) \to C$ are natural, so is $\psi\theta : A \to A$ given by $A \xrightarrow[\theta]{} L(A, A, A) \xrightarrow[\psi]{} A$.

It remains to prove that α is $\mathscr{P}$-natural in C, which, dually to (15), is equivalent to

$$\mathbf{t} = \mathbf{a}\mathbf{A}(1, \alpha)\mathbf{s}. \tag{16}$$

Now

$$\begin{aligned} \mathbf{a}\mathbf{A}(1, \alpha)\mathbf{s} &= \mathbf{a}\mathbf{A}(1, \alpha)\mathbf{b}\mathbf{B}(\beta, 1)\mathbf{t} \qquad \text{by (15)} \\ &= \mathbf{a}\mathbf{b}\mathbf{B}(1, T(1, \alpha))\mathbf{B}(\beta, 1)\mathbf{t} \qquad \text{by the naturality of } \mathbf{b} \\ &= \mathbf{B}(\beta, 1)\mathbf{B}(1, T(1, \alpha))\mathbf{t} \\ &= \mathbf{B}(\beta, 1)\mathbf{B}(T(1, \alpha), 1)\mathbf{t} \qquad \text{by the naturality of } \mathbf{t} \\ &= \mathbf{t} \qquad \text{by (3);} \end{aligned}$$

for the reader will see, on drawing a diagram, that the β and the $T(1, \alpha)$ involved in the second-last line are

$$T(C', A) \xrightarrow[\beta]{} T(C', S(T(C', A), C')) \xrightarrow[T(1,\alpha)]{} T(C', A).$$

Finally let us note some simplifications that appear when the functor F, like the forgetful functor from pointed sets or abelian groups to sets, is *isomorphism-reflecting:* by which we mean that a morphism f of $\mathscr{P}$ is an isomorphism whenever Ff is an isomorphism in $\mathscr{S}$. (F may well have this property without being faithful.)

With F isomorphism-reflecting, let $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$ be $\mathscr{P}$-categories and $(T, \mathbf{T}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ a $\mathscr{P}$-functor, and let $T : \mathscr{A} \to \mathscr{B}$ have an ordinary adjoint $S : \mathscr{B} \to \mathscr{A}$ with a, b, α, β as usual. Define $\mathbf{a} : \mathbf{A}(SB, A) \to \mathbf{B}(B, TA)$ by $\mathbf{a} = \mathbf{B}(\beta, 1)\mathbf{T}$; then $\mathbf{a}$ is natural, and is an isomorphism because $F\mathbf{a} = a$ is.

Moreover $\mathbf{aA}(\alpha, 1) = \mathbf{B}(\beta_T, 1)\mathbf{TA}(\alpha, 1) = \mathbf{B}(\beta_T, 1)\mathbf{B}(T\alpha, 1)\mathbf{T}$, by the naturality of $\mathbf{T}$; since $T\alpha \cdot \beta_T = 1$ by (3), $\mathbf{aA}(\alpha, 1) = \mathbf{T}$, so that the $\mathscr{P}$-functor structure that T receives from $\mathbf{a}$ is that it already has. Thus $(T, \mathbf{T})$ has a $\mathscr{P}$-adjoint $(S, \mathbf{S})$.

From Theorems 8 and 9 it now follows, if F is isomorphism-reflecting, that a $\mathscr{P}$-bifunctor T with an ordinary adjoint has a $\mathscr{P}$-adjoint, which is itself a $\mathscr{P}$-bifunctor; and that, if T is a $\mathscr{P}$-$\mathscr{P}$-bifunctor, so is the adjoint.

5. Tensor Products

Given $\mathscr{P}$ and $F : \mathscr{P} \to \mathscr{S}$, a $\mathscr{P}$-functor structure on $\mathscr{P}$ itself is a functor $\mathbf{P} : \mathscr{P}^* \times \mathscr{P} \to \mathscr{P}$ with $F\mathbf{P} = \mathscr{P} : \mathscr{P}^* \times \mathscr{P} \to \mathscr{S}$: that is, what we called in the introduction an internal Hom-functor.

As we shall see, $\mathbf{P}$ need not have an adjoint; but if it does, we write the adjoint as $\otimes$ and call it a tensor product. We have

$$a : \mathscr{P}(B \otimes C, A) \to \mathscr{P}(B, \mathbf{P}(C, A)),\ a^{-1} = b,$$
$$\alpha : \mathbf{P}(C, A) \otimes C \to A,\ \beta : B \to \mathbf{P}(C, B \otimes C).$$

We give some examples (with $\mathscr{P}$, $\mathbf{P}$ replaced by other letters, as $\mathscr{X}$, $\mathbf{X}$).

1. $\mathscr{S} =$ sets, $F = 1$, $\mathbf{S} = \mathscr{S}$, $A \otimes B = A \times B$.

2. $\mathscr{S}_0 =$ pointed sets, F forgetful, $\mathbf{S}_0(A, B) = \mathscr{S}_0(A, B)$ considered as a pointed set, $A \otimes B = A \times B/A \vee B$.

3. $\mathscr{G} =$ abelian groups, F forgetful, $\mathbf{G}(A, B) = \mathscr{G}(A, B)$ with usual group structure, $A \otimes B$ usual tensor product.

4. $\mathscr{H} =$ graded abelian groups, FA is the 0-component of A regarded as a set (and so F is not faithful), $\mathbf{H}(A, B)$ is the graded group of all homogeneous maps $A \to B$ of all degrees, $A \otimes B$ usual tensor product of graded groups.

5. $\mathscr{C} =$ chain complexes of abelian groups, $FA = Z_0 A$, the 0-cycles of A, regarded as a set; $\mathbf{C}(A, B) =$ complex of all homogeneous maps $A \to B$ of all degrees, $A \otimes B$ usual tensor product of chain complexes.

In the above examples, $\otimes$ is associative, commutative, and has a two-sided identity, to within coherent natural isomorphisms. The following examples are intended to show the degree of independence of these properties. We remark that we shall show below that a right identity for $\otimes$ is also a left identity, and so is unique (to within isomorphism).

Let $\mathscr{L}$ be the category of finite simplicial complexes, understood as finite sets with distinguished (spanning) subsets, any subset of a spanning set spanning, but not all the points in the complex necessarily belonging to spanning sets; and simplicial maps, understood as those set-maps that take spanning sets into spanning sets. Let F be the forgetful functor to sets (F forgets the distinguished subsets). Two different internal Hom-functors on $\mathscr{L}$ are given by: $\mathbf{L}(A, B) = \mathscr{L}(A, B)$ as a set, with the structure of a complex given by:

6. $\{f_1, \cdots, f_n\} \subset \mathscr{L}(A, B)$ spans if and only if $\{f_1 a, \cdots, f_n a\}$ spans in B for each $a \in A$ and also $f_1\bar{A} \cup \cdots \cup f_n\bar{A}$ spans in B, where $\bar{A}$ is the subset of A consisting of the points $a \in A$ with $\{a\}$ spanning.

7. $\{f_1, \cdots, f_n\} \subset \mathscr{L}(A, B)$ spans if and only if $f_1A \cup \cdots \cup f_nA$ spans in B.

Now let $\mathscr{K}$ be the full subcategory of $\mathscr{L}$ determined by the complexes A in which every singleton $\{a\}$ spans, and let F still be forgetful to sets. Define various internal Hom-functors by: $\mathbf{K}(A, B)$ is $\mathscr{K}(A, B)$ with $\{f_1, \cdots, f_n\} \subset \mathscr{K}(A, B)$ spanning if and only if:

8. Either $f_1A \cup \cdots \cup f_nA$ spans or $f_1 = f_2 = \cdots = f_n$.

9. $f_1C \cup \cdots \cup f_nC$ spans for each spanning set $C \subset A$.

10. $\{f_1a, \cdots, f_na\}$ spans for each $a \in A$.

11. $f_1 = f_2 = \cdots = f_n$.

12. No condition: *all* subsets span.

For a last example,

13. We take example 9 above, but consider only the full subcategory of $\mathscr{K}$ determined by the complexes with at least two points: which is closed under $\mathbf{K}$.

Once the examples are given, the verifications of the following statements are elementary, and we leave them to the reader. Notice that in examples 6–13 F is faithful, so that we are in the most favourable case.

First, that the internal Hom-functor need have no adjoint is shown by 8: $\mathbf{K}$ has no adjoint since $\mathbf{K}(A, -)$ does not preserve direct products. In all the other examples there is an adjoint $\otimes$.

In both examples 9 and 10, $\otimes$ is associative, commutative, and has a two-sided identity, to within coherent isomorphisms. This shows that the same $\mathscr{K}$ admits two essentially different tensor products, each with this degree of perfection.

In 13, $\otimes$ is still associative and commutative, but now has neither a left nor a right identity. The remaining examples all have at least a left identity. That this need not be unique, even when $\otimes$ is associative, is shown by 11, where every connected complex is a left identity for $\otimes$.

The left identity is, by a remark above, unique if there is also a right identity; but the existence of a unique left identity, even together with associativity, does not imply the existence of a right identity: this can be seen from 12. On the other hand, the existence of a right (and therefore a two-sided) identity does not imply associativity, as can be seen from 6.

Lastly, 7 shows that associativity together with a two-sided identity does not imply commutativity.

6. Associativity

Suppose we have a tensor product with the notation at the beginning of Section 5. Then any functor of many variables, made from the repeated use of $\otimes$ only, such as $A \otimes (B \otimes C)$, $(A \otimes B) \otimes C$, or $(A \otimes B) \otimes (C \otimes D)$, has a coadjoint made from applications of $\mathbf{P}$ and $\otimes$, the adjunction isomorphism being a composite of instances of a. For example,

$$\mathscr{P}(A \otimes (B \otimes C), D) \xrightarrow{a} \mathscr{P}(A, \mathbf{P}(B \otimes C, D)),$$

$$\mathscr{P}(A \otimes B) \otimes C, D) \xrightarrow{a} \mathscr{P}(A \otimes B, \mathbf{P}(C, D)) \xrightarrow{a} \mathscr{P}(A, \mathbf{P}(B, \mathbf{P}(C, D))),$$

$$\begin{aligned}\mathscr{P}([A \otimes B] \otimes [C \otimes D], E) &\xrightarrow{a} \mathscr{P}(A \otimes B, \mathbf{P}(C \otimes D, E)) \\ &\xrightarrow{a} \mathscr{P}(A, \mathbf{P}(B, \mathbf{P}(C \otimes D, E))).\end{aligned}$$

Throughout this section, when we speak of *the* coadjoint of such a functor, it is the coadjoint formed precisely in this way that we mean, with this particular adjunction that is a "power" of a.

Any natural transformation of functors of this kind is, by Theorem 4, reflected exactly by a natural transformation of their coadjoints. In particular, the natural isomorphisms $\rho : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, if any, are in 1–1 correspondence with the natural isomorphisms, if any,

$$\mathbf{a} : \mathbf{P}(B \otimes C, D) \to \mathbf{P}(B, \mathbf{P}(C, D)),$$

the correspondence being given by the commutativity of

$$\begin{tikzcd} \mathscr{P}(A \otimes (B \otimes C), D) \arrow[rr, "a"] \arrow[d, "\mathscr{P}(\rho,1)"'] & & \mathscr{P}(A, \mathbf{P}(B \otimes C, D)) \arrow[d, "\mathscr{P}(1,\mathbf{a})"] \\ \mathscr{P}((A \otimes B) \otimes C, D) \arrow[r, "a"'] & \mathscr{P}(A \otimes B, \mathbf{P}(C, D)) \arrow[r, "a"'] & \mathscr{P}(A, \mathbf{P}(B, \mathbf{P}(C, D))). \end{tikzcd}$$

Thus,

Theorem 10. *$\otimes$ is associative if and only if there is a $\mathscr{P}$-adjunction $\mathbf{a} : \mathbf{P}(B \otimes C, D) \to \mathbf{P}(B, \mathbf{P}(C, D))$ of $\otimes$ to $\mathbf{P}$.*

Now suppose that such a $\mathscr{P}$-adjunction $\mathbf{a}$ exists. Then $F\mathbf{a}$ is an ordinary adjunction of $\otimes$ to $\mathbf{P}$, which need not coincide with a. However a was, to begin with, just any adjunction of $\otimes$ to $\mathbf{P}$, and we now reject it in favor of

$F\mathbf{a}$, which we henceforth call a. Of course by so doing we alter the ρ that corresponds by (17) to the given $\mathbf{a}$; but the new ρ is determined entirely by $\mathbf{a}$.

The relationships between ρ, $\mathbf{a}$, α, and β, and an explicit formula for ρ, may be found if desired from (6), (7), and (8).

By Theorem 8, $\mathbf{P}$ and $\otimes$ are now $\mathscr{P}$-bifunctors, with $\mathscr{P}$-functor structures that we may denote by:

$$\mathbf{T} : \mathbf{P}(A, B) \to \mathbf{P}(\mathbf{P}(C, A), \mathbf{P}(C, B)),$$

$$\mathbf{S} : \mathbf{P}(A, B) \to \mathbf{P}(A \otimes C, B \otimes C).$$

Observe that $\mathbf{P}(B \otimes C, D)$ and $\mathbf{P}(B, \mathbf{P}(C, D))$ are both $\mathscr{P}$-functors in the argument D, with $\mathscr{P}$-functor structures $\mathbf{T}$ and $\mathbf{T}^2$ respectively; so that it makes sense to ask whether $\mathbf{a}$ is $\mathscr{P}$-natural in the argument D.

We now ask whether ρ, considered by itself, is coherent. By [*4*] this is equivalent to the commutativity of

$$\begin{tikzcd}
([A \otimes B] \otimes C) \otimes D \arrow[r, "\rho"] \arrow[d, "\rho\otimes 1"'] & {[A \otimes B] \otimes [C \otimes D]} \arrow[r, "\rho"] & A \otimes (B \otimes [C \otimes D]) \\
(A \otimes [B \otimes C]) \otimes D \arrow[rr, "\rho"'] & & A \otimes ([B \otimes C] \otimes D); \arrow[u, "1\otimes\rho"']
\end{tikzcd} \tag{18}$$

but the commutativity of (18) is equivalent by Theorem 4 to that of the coadjoint diagram, which, using the definition (17) of ρ as the adjoint of $\mathbf{a}$ and the naturality of a, turns out to be:

$$\begin{tikzcd}
\mathbf{P}(B \otimes [C \otimes D], E) \arrow[rr, "\mathbf{a}"] \arrow[d, "\mathbf{P}(\rho,1)"'] & & \mathbf{P}(B, \mathbf{P}(C \otimes D, E)) \arrow[d, "\mathbf{P}(1,\mathbf{a})"] \\
\mathbf{P}([B \otimes C] \otimes D, E) \arrow[r, "\mathbf{a}"'] & \mathbf{P}(B \otimes C, \mathbf{P}(D, E)) \arrow[r, "\mathbf{a}"'] & \mathbf{P}(B, \mathbf{P}(C, \mathbf{P}(D, E))).
\end{tikzcd} \tag{19}$$

Theorem 7 now gives at once:

THEOREM 11. *ρ is coherent if and only if* $\mathbf{a} : \mathbf{P}(C \otimes D, E) \to \mathbf{P}(C, \mathbf{P}(D, E))$ *is $\mathscr{P}$-natural in the argument E.*

If F is faithful the requirement of $\mathscr{P}$-naturality is vacuous, and ρ is always coherent. That $\mathbf{a}$ need not be $\mathscr{P}$-natural in general can be seen in example 4 of Section 5. Here the usual $\mathbf{a}$ *is* $\mathscr{P}$-natural, and the usual ρ is coherent. But if we replace $\mathbf{a}$ by $\bar{\mathbf{a}}$, where $\bar{\mathbf{a}}_n = (-1)^{n(n+1)/2}\mathbf{a}_n$ ($\mathbf{a}$ is a map of degree 0 of graded groups and $\mathbf{a}_n$ denotes its component in dimension n), then $a = \mathbf{a}_0$ is unchanged and ρ gets replaced by $\bar{\rho} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, where $\bar{\rho}((a \otimes b) \otimes c) = (-1)^{n(n+1)/2}\, a \otimes (b \otimes c)$, $n = \dim a$. It is easily verified that (18) is not commutative with $\bar{\rho}$ in place of ρ.

7. Left and Right Identities

Suppose we have a tensor product $\otimes$ as in Section 5, not necessarily associative. Then if the functor $F : \mathscr{P} \to \mathscr{S}$ is representable, which means that $FA \cong \mathscr{P}(I, A)$ for some I, (necessarily unique to within isomorphism by Theorem 1), I is a left identity for $\otimes$. For if $\phi : FA \to \mathscr{P}(I, A)$ is a natural isomorphism, we have natural isomorphisms

$$\mathscr{P}(A, B) = F\mathbf{P}(A, B) \xrightarrow[\phi]{} \mathscr{P}(I, \mathbf{P}(A, B)) \xrightarrow[b]{} \mathscr{P}(I \otimes A, B),$$

whence by Theorem 1 there is a natural isomorphism $\theta : I \otimes A \to A$ with

$$b\phi = \mathscr{P}(\theta, 1) : \mathscr{P}(A, B) \to \mathscr{P}(I \otimes A, B). \tag{20}$$

The converse conclusion, that if I is a left identity for $\otimes$ then $F \cong \mathscr{P}(I, -)$, certainly does not hold, for we have seen in Section 5 examples where there are many left identities, even when F is faithful and (not only is representable but) has an adjoint.

Now consider, independently of the existence of a left identity, the possibility that $\otimes$ has a right-identity.

Theorem 12. *$\otimes$ has a right identity I with $\tau : A \otimes I \cong A$ if and only if there is a natural isomorphism $\sigma : B \to \mathbf{P}(I, B)$. In this case I is also a left identity, and the natural isomorphism $\theta : I \otimes A \to A$ may be so chosen that we have the coherence property $\theta = \tau : I \otimes I \to I$.*

The first assertion follows from Theorem 4 when we observe that, in the following diagram, a and 1 are both adjunctions:

$$\begin{array}{ccc} \mathscr{P}(A \otimes I, B) & \xrightarrow{a} & \mathscr{P}(A, \mathbf{P}(I, B)) \\ {\scriptstyle \mathscr{P}(\tau,1)}\uparrow & & \uparrow{\scriptstyle \mathscr{P}(1,\sigma)} \\ \mathscr{P}(A, B) & \xrightarrow[1]{} & \mathscr{P}(A, B). \end{array}$$

Thus $\mathscr{P}(\tau, 1) = b\mathscr{P}(1, \sigma)$. (21)

Applying F to σ we get a natural isomorphism

$$\phi = F\sigma : FB \to F\mathbf{P}(I, B) = \mathscr{P}(I, B),$$

and so $\theta : I \otimes A \cong A$ as before. The naturality of σ contains as a special case the commutativity of

$$\begin{array}{ccc} B & \xrightarrow{\sigma} & \mathbf{P}(I, B) \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \mathbf{P}(1,\sigma)} \\ \mathbf{P}(I, B) & \xrightarrow[\sigma]{} & \mathbf{P}(I, \mathbf{P}(I, B)), \end{array}$$

which, since σ is an isomorphism, means

$$\mathbf{P}(1, \sigma) = \sigma : \mathbf{P}(I, B) \to \mathbf{P}(I, \mathbf{P}(I, B)). \tag{22}$$

Applying F to this gives

$$\mathscr{P}(1, \sigma) = \phi : \mathscr{P}(I, B) \to \mathscr{P}(I, \mathbf{P}(I, B)),$$

whence, in view of (20) and (21),

$$\tau = \theta : I \otimes I \to I. \tag{23}$$

Note that, using (6) and (7), we may express the relation of τ to σ in terms of α and β by the commutativity of:

$$\begin{tikzcd} A \otimes I \arrow[r, "\sigma\otimes 1"] \arrow[dr] & \mathbf{P}(I, A) \otimes I \arrow[d, "\alpha"] \\ & A \end{tikzcd}, \qquad \begin{tikzcd} A \arrow[r, "\beta"] \arrow[dr, "\sigma"'] & \mathbf{P}(I, A \otimes I) \arrow[d, "\mathbf{P}(1,\tau)"] \\ & \mathbf{P}(I, A). \end{tikzcd} \tag{24}$$

Finally we make some remarks about uniqueness. Given $\mathscr{P}, F$, and $\mathbf{P}$, then $\otimes$ as the adjoint of $\mathbf{P}$ is determined to within a natural isomorphism, and for a fixed choice of $\otimes$ the adjunction a is determined to within postmultiplication by $\mathscr{P}(\kappa, 1)$ where κ is a natural automorphism of $\otimes$: all of which is as much as one could expect. Now if there are two natural isomorphisms $\phi : FA \to \mathscr{P}(I, A)$, $\phi' : FA \to \mathscr{P}(I', A)$, we must, by Theorem 1, have $\phi' = \mathscr{P}(\lambda, 1)\phi$ where $\lambda : I' \to I$ is an isomorphism, and then $\theta' : I' \otimes A \to A$ is $\theta(\lambda \otimes 1)$; for a fixed choice of I, ϕ is still indeterminate by $\mathscr{P}(\lambda, 1)$, where now λ is an automorphism of I. Again this is all we could expect. But now if there are two natural isomorphisms $\sigma : A \to \mathbf{P}(I, A)$, $\sigma' : A \to \mathbf{P}(I', A)$, and if $\phi = F\sigma$ and $\phi' = F\sigma'$, we get as above an isomorphism $\lambda : I' \to I$, but we *cannot* assert that $\sigma' = \mathbf{P}(\lambda, 1)\sigma$ in the nonfaithful case. For in example 4 of Section 5, with I the graded group which is infinite cyclic in dimension 0 and 0 in other dimensions, let σ be the usual isomorphism and define

$$\sigma' : A \to \mathbf{H}(I, A) \quad \text{by} \quad \sigma'_n = (-1)^n \sigma_n .$$

Then

$$\phi' = F\sigma' = \sigma'_0 = \sigma_0 = F\sigma = \phi.$$

We return to this question in the next section.

8. Left and Right Identities with Associativity

We now return to the associative $\otimes$ of Section 6. Suppose F is representable, with $\phi : FA \cong \mathscr{P}(I, A)$ giving $\theta : I \otimes A \to A$ as in Section 7.

THEOREM 13. *The coherence condition expressed by the commutativity of*

$$\begin{tikzcd}
(I \otimes B) \otimes C \arrow[rr, "\rho"] \arrow[dr, "\theta \otimes 1"'] & & I \otimes (B \otimes C) \arrow[dl, "\theta"] \\
& B \otimes C &
\end{tikzcd} \tag{25}$$

is always satisfied.

Consider the adjunctions of $(I \otimes -) \otimes C$ to $\mathbf{P}(C, -)$, of $I \otimes (- \otimes C)$ to $\mathbf{P}(C, -)$, and of $- \otimes C$ to $\mathbf{P}(C, -)$, given respectively by:

$$\mathscr{P}((I \otimes B) \otimes C, D) \xrightarrow{a} \mathscr{P}(I \otimes B, \mathbf{P}(C, D)) \xrightarrow{a} \mathscr{P}(I, \mathbf{P}(B, \mathbf{P}(C, D)))$$
$$\xrightarrow{\phi^{-1}} \mathscr{P}(B, \mathbf{P}(C, D)),$$

$$\mathscr{P}(I \otimes (B \otimes C), D) \xrightarrow{a} \mathscr{P}(I, \mathbf{P}(B \otimes C, D)) \xrightarrow{\phi^{-1}} \mathscr{P}(B \otimes C, D)$$
$$\xrightarrow{a} \mathscr{P}(B, \mathbf{P}(C, D)),$$

$$\mathscr{P}(B \otimes C, D) \xrightarrow{\qquad\qquad a \qquad\qquad} \mathscr{P}(B, \mathbf{P}(C, D)).$$

Using the definitions of ρ and θ, it follows easily that the diagram coadjoint to (25) is

$$\begin{tikzcd}
\mathbf{P}(C, D) & & \mathbf{P}(C, D) \arrow[ll, "1"'] \\
& \mathbf{P}(C, D) \arrow[ul, "1"] \arrow[ur, "1"'] &
\end{tikzcd} ,$$

which commutes.

Now suppose I is a right identity for $\otimes$, with the notation as in Section 7.

THEOREM 14. *The coherence condition expressed by the commutativity of*

$$\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "\rho"] \arrow[dr, "\tau"'] & & A \otimes (B \otimes I) \arrow[dl, "1 \otimes \tau"] \\
& A \otimes B &
\end{tikzcd} \tag{26}$$

holds if and only if $\sigma : A \to \mathbf{P}(I, A)$ *is* $\mathscr{P}$*-natural with respect to the identical* $\mathscr{P}$*-functor structure on* A *and the* $\mathscr{P}$*-functor structure* $\mathbf{T}$ *on* $\mathbf{P}(I, A)$.

We pass from (22) to the coadjoint diagram: using the definitions of ρ and τ as adjoints of $\mathbf{a}$ and σ, and the naturality of a, and arguing as in the proof that precedes Theorem 11, we easily see that the commutativity of (26) is equivalent to that of

$$
\begin{tikzcd}
\mathbf{P}(B \otimes I, C) \arrow[rr, "\mathbf{a}"] & & \mathbf{P}(B, \mathbf{P}(I, C)) \\
& \mathbf{P}(B, C) \arrow[ul, "\mathbf{P}(\tau,1)"] \arrow[ur, "\mathbf{P}(1,\sigma)"'] &
\end{tikzcd}
\;;
$$

comparing this with (21), and using Theorem 7, we get the desired result.

THEOREM 15. *The coherence condition expressed by the commutativity of*

$$
\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\rho"] \arrow[dr, "\tau\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes\theta"] \\
& A \otimes B &
\end{tikzcd}
\tag{27}
$$

is equivalent to the $\mathscr{P}$-naturality in the argument C of the composition $\sigma^{-1}\mathbf{a}$:

$$\mathbf{P}(I \otimes B, C) \xrightarrow[\mathbf{a}]{} \mathbf{P}(I, \mathbf{P}(B, C)) \xrightarrow[\sigma^{-1}]{} \mathbf{P}(B, C)$$

and so is satisfied if both $\mathbf{a}$ *and* σ *are $\mathscr{P}$-natural.*

THEOREM 16. *The necessary and sufficient conditions for the full coherence of ρ, θ, and τ is that both* $\mathbf{a}$ *and σ be $\mathscr{P}$-natural.*

Admitting Theorem 15, Theorem 16 follows from [*4*], where the conditions for coherence are shown to be (18), (23), (25), (26), (27).[2]

We turn to the proof of Theorem 15. All the functors in (27) have coadjoints, and the coadjoint diagram to (27) is easily found to be

$$
\begin{tikzcd}
\mathbf{P}(I, \mathbf{P}(B, C)) & & \mathbf{P}(I \otimes B, C) \arrow[ll, "\mathbf{a}"'] \\
& \mathbf{P}(B, C). \arrow[ul, "\sigma"] \arrow[ur, "\mathbf{P}(\theta,1)"'] &
\end{tikzcd}
$$

On the one hand, the diagram obtained by applying F to this does in fact commute, by (20). On the other hand, $\mathbf{P}(\theta, 1)$ in this diagram is $\mathscr{P}$-natural in C by Theorem 8. Theorem 15 therefore follows once we have proved:

[2] *Note added in proof*: Since writing this the author has shown that (18) and (27) imply (23), (25), and (26) : see G. M. KELLY, On MacLane's conditions for coherence of natural associativities, commutativities, etc. *J. Algebra,* in press.

THEOREM 17. *Let $(\mathscr{P}, F)$ be a semiconcrete category with an internal* Hom-*functor* $\mathbf{P}$, *which need not have an adjoint. Let $\mathscr{P}$ contain an object I such that there is a natural isomorphism $\sigma : A \to \mathbf{P}(I, A)$, giving a natural isomorphism $\phi = F\sigma : FA \to \mathscr{P}(I, A)$. Let T, $T' : \mathscr{P} \to \mathscr{P}$ be functors with $\mathscr{P}$-adjoints $S, S' : \mathscr{P} \to \mathscr{P}$, and let $\lambda, \mu : T \to T'$ be transformations which are $\mathscr{P}$-natural with respect to the $\mathscr{P}$-functor structures that T and T' receive from the given $\mathscr{P}$-adjunctions. Let λ and μ have adjoints λ^*, $\mu^* : S' \to S$. Then the following conditions are equivalent:*

(a) $$F\lambda = F\mu : FT \to FT',$$

(b) $$\lambda_I^* = \mu_I^* : S'I \to SI;$$

and each implies that $\lambda = \mu$.

The equivalence of (a) and (b) follows from the diagram

$$\begin{tikzcd}
\mathscr{P}(SI, A) \arrow[r, "a"] \arrow[d, "\mathscr{P}(\lambda_I^*, 1)"'] & \mathscr{P}(I, TA) \arrow[d, "\mathscr{P}(1, \lambda)"] & FTA \arrow[l, "\phi"'] \arrow[d, "F\lambda"] \\
\mathscr{P}(S'I, A) \arrow[r, "a'"'] & \mathscr{P}(I, T'A) & FT'A \arrow[l, "\phi"]
\end{tikzcd}$$

and a similar diagram for μ. The conclusion $\lambda = \mu$ follows from the diagram

$$\begin{tikzcd}
\mathbf{P}(SI, A) \arrow[r, "\mathbf{a}"] \arrow[d, "\mathbf{P}(\lambda_I^*, 1)"'] & \mathbf{P}(I, TA) \arrow[d, "\mathbf{P}(1, \lambda)"] & TA \arrow[l, "\sigma"'] \arrow[d, "\lambda"] \\
\mathbf{P}(S'I, A) \arrow[r, "\mathbf{a}'"'] & \mathbf{P}(I, T'A) & T'A \arrow[l, "\sigma"]
\end{tikzcd}$$

and a similar diagram for μ.

We now return to the question of uniqueness discussed at the end of Section 7. We saw there that $\otimes$, a, θ, ϕ were essentially determined by $\mathscr{P}$, F, and $\mathbf{P}$, but that σ (and so τ) were not; neither (as examples above have shown) are $\mathbf{a}$ and ρ, when they exist. However, for a fixed choice of $\mathbf{a}$, both σ and τ *are* essentially determined if we are to have coherence of ρ, θ, and τ. For then σ is $\mathscr{P}$-natural, and so, since its range and its domain both have $\mathscr{P}$-adjoints, it is, by Theorem 17, determined by $F\sigma$, $= \phi$. The extent to which $\mathbf{a}$ itself is undetermined will be discussed in the second part of this paper.

Finally, note that Theorem 17 allows us to give a proof of Theorem 16 independently of the results of [*4*]. $V(A_1, \cdots, A_n)$ is some functor of $A_1, \cdots, A_n$, made from applications of $\otimes$ alone, suitably bracketed; and such a V surely has a $\mathscr{P}$-coadjoint. λ^* and $\mu^* : V \to V$ are natural automorphisms made from ρ, θ, τ and their inverses; and under our hypotheses they are $\mathscr{P}$-natural. The desired result $\lambda^* = \mu^*$ therefore follows from $\lambda_I^* = \mu_I^*$.

But the assertion $\lambda_I^* = \mu_I^*$ is soon seen, by the use of (25) etc., to be the same kind of assertion as $\lambda^* = \mu^*$, but with $(n - 1)$ variables instead of n. The result then follows by induction.

References

1. Kelly, G. M., Complete Functors in Homology. II. *Proc. Cambridge Phil. Soc.*, in press.
2. Kan, D. M., Adjoint functors. *Trans. Am. Math. Soc.* **87** (1958), 294-329.
3. MacLane, S., "Categorical Algebra." Boulder Colloquium Lectures, 1963.
4. MacLane, S., Natural associativity and commutativity. *Rice Univ. Stud.* **49** (1963), 28-46.